THE LARGEST SINGLETONS OF SET PARTITIONS

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Dedicated to L.C. Hsu, on the occasion of his ninetieth birthday

Abstract. Recently, Deutsch and Elizalde studied the largest and the smallest fixed points of permutations. Motivated by their work, we consider the analogous problems in set partitions. Let $A_{n,k}$ denote the number of partitions of $\{1, 2, \dots, n+1\}$ with the largest singleton $\{k+1\}$ for $0 \leq k \leq n$. In this paper, several explicit formulas for $A_{n,k}$, involving a Dobinski-type analog, are obtained by algebraic and combinatorial methods, many combinatorial identities involving $A_{n,k}$ and Bell numbers are presented by operator methods, and congruence properties of $A_{n,k}$ are also investigated. It will been showed that the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ (mod p) are periodic for any prime p, and contain a string of $p-1$ consecutive zeroes. Moreover their minimum periods are conjectured to be $N_p = \frac{p^p-1}{p-1}$ for any prime p.

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1. Introduction

A *partition* of a set $[n] = \{1, 2, \dots, n\}$ is a collection of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. It is well known that the number of partitions of $[n]$ with exactly k blocks is the Stirling number of the second kind $S(n, k)$ [17, A008267] and the total number of partitions of $[n]$ is the n-th Bell number B_n [16], beginning with $(B_n)_{n\geq 0} = (1, 1, 2, 5, 15, 52, 203, \dots)$ [17, A000110] and having the exponential generating function [19]

$$B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!} = \exp(e^x - 1). \tag{1.1}$$

Differentiating (1.1) gives $B'(x) = e^x B(x)$, which leads to

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{1.2}$$

A *singleton* of a partition is a block containing just one element. If $\{k\}$ is a singleton of a partition, we denote it by k for short. The number of partitions of $[n]$ without singletons is counted by V_n beginning with $(V_n)_{n\geq 0} = (1, 0, 1, 1, 4, 11, 41, 162, \dots)$ [17, A000296], and having the exponential generating function

$$V(x) = \sum_{n\geq 0} V_n \frac{x^n}{n!} = \exp(e^x - x - 1). \tag{1.3}$$

Bernhart [2] has given a combinatorial interpretation for the relation $B_n = V_n + V_{n+1}$ which can also be obtain from $B(x) = V(x) + V'(x)$. By (1.1) and (1.3), one can deduce that

$$B_n = \sum_{j=0}^{n} \binom{n}{j} V_j \quad \text{and} \quad V_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_j.$$

Recently, Deutsch and Elizalde [5] study the largest and the smallest fixed points of permutations. Motivated by their work, we consider the analogous problems in set partitions. Let $A_{n,k}$ denote the number of partitions of $[n+1]$ with the largest singleton $k+1$. Clearly,

$$A_{n,0} = V_n \quad \text{and} \quad A_{n,n} = B_n.$$

This paper is organized as follows. In the next section, we find several explicit formulas for $A_{n,k}$, involving a Dobinski-type analog, by algebraic and combinatorial methods. In the section 3, we obtain many combinatorial identities involving $A_{n,k}$ and Bell numbers B_n by operator methods. In the last section, we consider the congruence properties of $A_{n,k}$ and Bell numbers B_n, find that the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ (modulo p) are periodic for any prime p and contain a string of $p-1$ consecutive zeroes. We also conjecture that their minimum periods are $N_p = \frac{p^p-1}{p-1}$ for any prime p.

2. The explicit formulas for $A_{n,k}$

It follows from the definition that

$$A_{n,k} = V_n + \sum_{j=0}^{k-1} A_{n-1,j}, \tag{2.1}$$

since by removing the largest singleton $k+1$ of a partition of $[n+1]$ containing singletons, we get a partition of $\{1,\dots,k,k+2,\dots,n+1\}$ whose largest singleton (if any) is less than $k+1$.

In (2.1), if we replace k by $k-1$, then by subtraction we obtain a recurrence for $n,k\geq 1$,

$$A_{n,k} = A_{n,k-1} + A_{n-1,k-1}. \tag{2.2}$$

Table 1 shows the values of $A_{n,k}$ for small n and k. It should be noticed that $\{A_{n+k,k}\}_{n\geq k\geq 1}$ is just the Aitken's array [17, A011971]. We point out that it is possible to give a direct combinatorial proof of the recurrence (2.2) from the definition of the $A_{n,k}$. Indeed, given a partition π of $[n+1]$ with the largest singleton $k+1$, if k is also a singleton, delete the singleton $k+1$ and subtracting one from all the entries large than $k+1$, we obtain a partition of $[n]$ with the largest singleton k; if k is not a singleton, exchange k and $k+1$, we obtain a partition of $[n+1]$ with the largest singleton k.

n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	1	2					
3	1	2	3	5				
4	4	5	7	10	15			
5	11	15	20	27	37	52		
6	41	52	67	87	114	151	203	
7	162	203	255	322	409	523	674	877

Table 1. The values of $A_{n,k}$ for n and k up to 7.

When $k=1$, (2.2) produces a new setting for Bell numbers, namely $A_{n+1,1} = B_n$. A simple combinatorial proof reads: given a partition π of $[n+2]$ with the largest singleton 2, if 1 is also a singleton, delete the two singletons $1,2$ and subtracting two from all the entries large than 2, we obtain a partition of $[n]$ without singletons; if 1 is not a singleton, break the block containing 1 into singletons (more than one), then delete the two singletons $1,2$ and subtracting two from all the entries large than 2, we obtain a partition of $[n]$ with singletons.

Lemma 2.1. *The bivariate exponential generating function for $A_{n+k,k}$ is given by*

$$A(x,y) = \sum_{n,k\geq 0} A_{n+k,k}\frac{x^n}{n!}\frac{y^k}{k!} = \exp(e^{x+y} - x - 1).$$

Proof. Define

$$A_k(x) = \sum_{n\geq 0} A_{n+k,k}\frac{x^n}{n!}.$$

Clearly, $A_0(x) = \exp(e^x - x - 1)$ and $A_1(x) = \exp(e^x - 1)$. From (2.2), one can derive that

$$A_k(x) = A_{k-1}(x) + A'_{k-1}(x).$$

Let $\mathcal{D}$ denote the derivative with respect to x, we have

$$A_k(x) = (1+\mathcal{D})A_{k-1}(x) = (1+\mathcal{D})^k A_0(x).$$

Then

$$\begin{aligned} A(x,y) &= \sum_{k\geq 0} A_k(x)\frac{y^k}{k!} = \sum_{k\geq 0}\frac{y^k(1+\mathcal{D})^k}{k!}A_0(x) \\ &= e^{y+y\mathcal{D}}A_0(x) = e^y e^{y\mathcal{D}}A_0(x) = e^y A_0(x+y) \\ &= \exp(e^{x+y} - x - 1). \end{aligned}$$

This complete the proof. □

The general formula for the Bell polynomial $B_k(x) = \sum_{j=0}^k S(k,j)x^j$ states that

$$B_k(x) = e^{-x}\sum_{m\geq 0}\frac{m^k x^m}{m!},$$

which, when $x = 1$, produces the Dobinski's formula [16] for Bell numbers

$$B_k = \frac{1}{e}\sum_{m\geq 0}\frac{m^k}{m!}.$$

Analogously, we can derive a Dobinski-type formula for $A_{n+k,k}$.

Theorem 2.2. *For any integers $n, k \geq 0$, there holds*

$$A_{n+k,k} = \frac{1}{e}\sum_{m=0}^{\infty}\frac{m^k(m-1)^n}{m!}. \tag{2.3}$$

Proof. By Lemma 2.1, one has

$$\begin{aligned} A(x,y) &= \exp(e^{x+y} - x - 1) \\ &= e^{-x-1}\sum_{m\geq 0}\frac{e^{(x+y)m}}{m!} \\ &= e^{-1}\sum_{m\geq 0}\frac{1}{m!}\sum_{n\geq 0}\frac{(m-1)^n x^n}{n!}\sum_{k\geq 0}\frac{m^k y^k}{k!} \\ &= e^{-1}\sum_{n,k\geq 0}\frac{x^n}{n!}\frac{y^k}{k!}\sum_{m\geq 0}\frac{m^k(m-1)^n}{m!}, \end{aligned}$$

which leads to (2.3) by comparing the coefficients of $\frac{x^n}{n!}\frac{y^k}{k!}$. □

Remark 2.3. *According to the Dobinski-type formula for $A_{n+k,k}$, one can deduce the column generating function $A_k(x) = V(x)B_k(e^x)$. By attracting the coefficient of $\frac{y^k}{k!}$ from $A(x, y)$, one can also find $A_k(x) = e^{-x}\sum_{n\geq 0} B_{n+k}\frac{x^n}{n!} = e^{-x}\mathcal{D}^k B(x)$. Then one has the relation for Bell polynomials $\mathcal{D}^k B(x) = B(x)B_k(e^x)$.*

Theorem 2.4. *For any integers $n, m, k \geq 0$, there hold*

$$A_{n+m,m} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j}, \tag{2.4}$$

$$A_{n+m+k,m+k} = \sum_{j=0}^{m}\binom{m}{j}A_{n+k+j,k}. \tag{2.5}$$

Proof. Note that $A(x, y) = B(x + y)e^{-x}$ and $\frac{\partial^k}{\partial y^k}A(x, y) = A_k(x + y)e^y$ from Lemma 2.1, by equating the coefficients of $\frac{x^n y^m}{n!m!}$ in the resulting series, one can easily deduce (2.4)-(2.5). Here we provide a combinatorial proof.

(1) Let $\mathbb{S}$ denote the set of partitions of $[n + m + 1]$ containing at least the singleton $m + 1$, Clearly, $|\mathbb{S}| = B_{m+n}$. Let $\mathbb{S}_i$ be the subset of $\mathbb{S}$ containing another singleton $m + i + 1$ for $1 \leq i \leq n$. Set $\overline{\mathbb{S}}_i = \mathbb{S} - \mathbb{S}_i$, then $\bigcap_{i=1}^{n} \overline{\mathbb{S}}_i$, counted by $A_{n+m,m}$, is just the set of partitions of $[n + m + 1]$ with the largest singleton $m + 1$. For any nonempty $(n - j)$-subset $\mathbb{A} \in [n]$, $\bigcap_{i\in\mathbb{A}} \mathbb{S}_i$, counted by B_{m+j}, is the set of partitions of $[n+m+1]$ containing at least the number $n-j+1$ of singletons $m+1$ and $m+i+1$ for all $i \in \mathbb{A}$. By the Inclusion-Exclusion principle, we have

$$\begin{aligned}
|\bigcap_{i=1}^{n}\overline{\mathbb{S}}_i| &= |\mathbb{S} - \bigcup_{i=1}^{n}\mathbb{S}_i| \\
&= |\mathbb{S}| + \sum_{j=0}^{n-1}(-1)^{n-j}\binom{n}{j}|\bigcap_{i\in\mathbb{A},|\mathbb{A}|=n-j}\mathbb{S}_i| \\
&= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j},
\end{aligned}$$

which proves (2.4).

(2) A partition π of $[n + m + k + 1]$ with the largest singleton $m + k + 1$ can be obtained as follows. Suppose that π has exactly $m-j$ singletons in $\{k+1, \dots, k+m\}$, there are $\binom{m}{j}$ ways to do this, so the remainder j elements in $\{k+1, \dots, k+m\}$ can not be singletons in π. These j elements can be regarded as the roles that greater than $m + k + 1$, there are $A_{n+k+j,k}$ ways to produce a partition π' of the remainder $n + k + j + 1$ elements with the largest singleton $m + k + 1$, then π' together with the $m - j$ singletons forms the desired partition π. Thus there are $\binom{m}{j}A_{n+k+j,k}$ of such partitions. Summing up all the possible cases yields (2.5). □

The cases $k = 0$ and $k = 1$ in (2.5) produce

Corollary 2.5. *For any integers $n, m \geq 0$, there hold*

$$A_{n+m,m} = \sum_{j=0}^{m}\binom{m}{j}V_{n+j},$$

$$A_{n+m+1,m+1} = \sum_{j=0}^{m}\binom{m}{j}B_{n+j}. \tag{2.6}$$

Remark 2.6. *The case $m := m+1$ in (2.4), together with (2.6), produces another identity for Bell numbers*

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j+1} \quad = \quad \sum_{j=0}^{m}\binom{m}{j}B_{n+j}.$$

Spivey [18] finds a generalized recurrence for Bell numbers

$$B_{n+k} = \sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}B_r S(k,j) j^{n-r},$$

and gives it a simple combinatorial proof. This recurrence has been generalized by Belbachir and Mihoubi [1], Gould and Quaintance [9]. We also have a similar formula for $A_{n+k,k}$.

Theorem 2.7. *For any integers $n, k \geq 0$, there hold*

$$A_{n+k,k} \quad = \quad \sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}V_r S(k,j) j^{n-r}, \tag{2.7}$$

$$A_{n+k,k} \quad = \quad \sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}B_r S(k,j)(j-1)^{n-r}. \tag{2.8}$$

Proof. Note that $A_k(x) = V(x)B_k(e^x)$ and $A_k(x) = B(x)B_k(e^x)e^{-x}$ from Remark 2.3, by equating the coefficients of $\frac{x^n}{n!}$ in the resulting series, one can easily deduce (2.7)-(2.8). Here we provide a combinatorial proof.

For the set $[n+k+1]$, one can count the number of ways to partition these $n+k+1$ elements in the following manners.

(1) Partition the set $[k]$ into exactly j blocks, there are $S(k,j)$ ways to do this. Choose an r-subset from the set $\{k+2,\ldots,n+k+1\}$ to be partitioned into new blocks, and distribute the remainder $n-r$ elements among the j blocks formed from the set $[k]$. There are $\binom{n}{r}$ ways to choose the r elements, V_r ways to partition them into new blocks without singletons, and j^{n-r} ways to distribute the remainder $n-r$ elements among the j blocks. Thus there are $\binom{n}{r}V_r S(k,j)j^{n-r}$ of such partitions. Note that $k+1$ is always a singleton, summing over all possible values of j and r produces all ways to partition the set $[n+k+1]$ with the largest singleton $k+1$. This gives a proof of (2.7).

(2) Partition the set $[k]$ into exactly j blocks $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_j$ and assume that $1 \in \mathbb{S}_1$, there are $S(k,j)$ ways to do this. Choose an r-subset $\mathbb{T}_r$ from the set $\{k+2,\ldots,n+k+1\}$ to be partitioned into new blocks, and distribute the remainder $n-r$ elements among the $j-1$ blocks $\mathbb{S}_2,\ldots,\mathbb{S}_j$, then merge all the singletons formed from the r-subset $\mathbb{T}_r$ (having been partitioned) into $\mathbb{S}_1$ to form one block. There are $\binom{n}{r}$ ways to choose the r elements, B_r ways to partition them into new blocks, and $(j-1)^{n-r}$ ways to distribute the remainder $n-r$ elements among the $j-1$ blocks. Thus there are $\binom{n}{r}B_r S(k,j)(j-1)^{n-r}$ of such partitions. Note that $k+1$ is the largest singleton, summing over all possible values of j and r produces all ways to partition the set $[n+k+1]$ with the largest singleton $k+1$. This gives a proof of (2.8). □

3. Identities involving $A_{n,k}$ and Bell numbers B_n

Theorem 3.1. *For any integer $n \geq 0$ and any indeterminant y, there hold*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k}(y+1)^k = \sum_{k=0}^{n}\binom{n}{k}y^k B_k, \tag{3.1}$$

or equivalently

$$\sum_{k=0}^{n}\binom{n}{k}A_{n,k}y^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(y+1)^kB_k. \tag{3.2}$$

Proof. Note that $A(-x,x(y+1))=B(xy)e^x$ and $A(x,xy)=B(x(y+1))e^{-x}$ from Lemma 2.1, by equating the coefficients of $\frac{x^n}{n!}$ in the resulting series, one can easily deduce (3.1)-(3.2). Also (3.2) can be obtained from (3.1) by setting $y:=-y-1$. One can be asked to give a combinatorial proof for these two identities. □

Corollary 3.2. *For any integer $n\geq 0$, there hold*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k} = 1, \tag{3.3}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^kA_{n,k} = B_{n+1}. \tag{3.4}$$

Proof. The case $y=0$ in (3.1) yields (3.3). The case $y=1$ in (3.1), together with (1.2), yields (3.4). □

Corollary 3.3. *For any integer $n\geq 0$ and any indeterminant y, there hold*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k}B_{k+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}B_kB_k(y), \tag{3.5}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_kB_{k+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}A_{n,k}B_k(y). \tag{3.6}$$

Proof. This is an equivalent form of Theorem 3.1. Define a linear (invertible) transformation

$$L_1(y^k)=B_k(y),\quad (k=0,1,2,\dots).$$

It is well known that $B_k(y)$ satisfies the relation

$$B_{n+1}(y)=y\sum_{k=0}^{n}\binom{n}{k}B_k(y).$$

Then we have

$$yL_1((y+1)^n)=y\sum_{k=0}^{n}\binom{n}{k}L_1(y^k)=y\sum_{k=0}^{n}\binom{n}{k}B_k(y)=B_{n+1}(y).$$

Hence (3.5) and (3.6) follow by acting yL_1 on the two sides of (3.1) and (3.2) respectively. □

Similarly, if define another linear transformation

$$L_2(y^k)=\binom{y}{k},\quad (k=0,1,2,\dots),$$

by the Vandermonde's convolution identity

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n},$$

we have

$$L_2((y+1)^n)=\sum_{k=0}^{n}\binom{n}{k}L_2(y^k)=\binom{y+n}{n}.$$

Then acting L_2 on the two sides of (3.1) and (3.2) leads respectively to another equivalent form of Theorem 3.1.

Corollary 3.4. *For any integer $n \geq 0$ and any indeterminant y, there hold*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y+k}{k}A_{n,k} &= \sum_{k=0}^{n}\binom{n}{k}\binom{y}{k}B_k,\\ \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y+k}{k}B_k &= \sum_{k=0}^{n}\binom{n}{k}\binom{y}{k}A_{n,k}.\end{aligned}$$

With the Bell umbra $\mathbf{B}$ [7, 14, 15], given by $\mathbf{B}^n = B_n$, (1.2) may be written as $\mathbf{B}^{n+1} = (\mathbf{B}+1)^n$. By (2.4), $A_{n,k}$ can be written umbrally as

$$A_{n,k} = \mathbf{B}^k(\mathbf{B}-1)^{n-k}.$$

Setting $y = \frac{y}{1-y}$ in (3.1) and (3.2), and multiplying $y^m(1-y)^n$ by their two sides, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}A_{n,k}y^m(y-1)^{(n+m-k)-m} &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{m+k}(y-1)^{(n+m)-(m+k)}B_k,\\ \sum_{k=0}^{n}\binom{n}{k}y^m(y-1)^{(n+m-k)-m}B_k &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{m+k}(y-1)^{(n+m)-(m+k)}A_{n,k},\end{aligned}$$

which, when $y = \mathbf{B}$, produce another two identities.

Corollary 3.5. *For any integers $n, m \geq 0$, there hold*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n+m,m+k}B_k &= \sum_{k=0}^{n}\binom{n}{k}A_{n+m-k,m}A_{n,k},\\ \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n+m,m+k}A_{n,k} &= \sum_{k=0}^{n}\binom{n}{k}A_{n+m-k,m}B_k.\end{aligned}$$

Theorem 3.6. *For any integers $n, k \geq 0$ and any indeterminant y, there hold*

$$\sum_{j=0}^{n}\binom{n}{j}A_{k+j,k}(y+1)^{n-j} = \sum_{j=0}^{n}\binom{n}{j}B_{k+j}y^{n-j}, \tag{3.7}$$

$$\sum_{j=0}^{n}\binom{n}{j}A_{k+j,k}B_{n-j+1}(y) = y\sum_{j=0}^{n}\binom{n}{j}B_{k+j}B_{n-j}(y), \tag{3.8}$$

$$\sum_{j=0}^{n}\binom{n}{j}\binom{y+n-j}{n-j}A_{k+j,k} = \sum_{j=0}^{n}\binom{n}{j}\binom{y}{n-j}B_{k+j}. \tag{3.9}$$

Proof. Note that $A(x,t)e^{x(y+1)} = B(x+t)e^{xy}$ from Lemma 2.1, by equating the coefficients of $\frac{x^n t^k}{n!k!}$ in the resulting series, one can easily deduce (3.7). (3.8) and (3.9) can be followed respectively by acting yL_1 and L_2 on the two sides of (3.7). Here we provide a combinatorial proof for (3.7).

Let $\mathbb{X}_{n,k} = \bigcup_{j=0}^{n}\mathbb{X}_{n,k,j}$ and $\mathbb{X}_{n,k,j}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-j)$-subset of $[k+2, n+k+1] = \{k+2, \dots, n+k+1\}$, and each element of $\mathbb{S}$ has weight 1 or y; In other words, each element of $\mathbb{S}$ has weight $1+y$;
- π is a partition of the set $[n+k+1] - \mathbb{S}$ with the largest singleton $k+1$, and each element of $[n+k+1] - \mathbb{S}$ has weight 1.

Let $\mathbb{Y}_{n,k} = \bigcup_{j=0}^{n} \mathbb{Y}_{n,k,j}$ and $\mathbb{Y}_{n,k,j}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-j)$-subset of $[k+2, n+k+1]$ and each element of $\mathbb{S}$ has weight y;
- π is a partition of the set $[n+k+1] - \mathbb{S}$ such that $k+1$ must be a singleton, and each element of $[n+k+1] - \mathbb{S}$ has weight 1.

The weight of $(\pi, \mathbb{S})$ is defined to be the product of the weight of each element of $[n+k+1]$. Clearly, the weights of $\mathbb{X}_{n,k}$ and $\mathbb{Y}_{n,k}$ are counted respectively by the left and right sides of (3.7).

Given any pair $(\pi, \mathbb{S}) \in \mathbb{X}_{n,k}$, $\mathbb{S}$ can be partitioned into two parts $\mathbb{S}_1$ and $\mathbb{S}_2$ such that each element of $\mathbb{S}_1$ has weight y and each element of $\mathbb{S}_2$ has weight 1. Regard each element of $\mathbb{S}_2$ as a singleton, together with π, we obtain a partition π_1 of $[n+k+1] - \mathbb{S}_1$ such that $k+1$ is a singleton. Then the pair $(\pi_1, \mathbb{S}_1)$ lies in $\mathbb{Y}_{n,k}$.

Conversely, for any pair $(\pi_1, \mathbb{S}_1) \in \mathbb{Y}_{n,k}$, let $\mathbb{S}$ denote the union of $\mathbb{S}_1$ and the singletons of π_1 greater than $k+1$, then π_1 can be partitioned into two parts π and π' such that π is a partition of $[n+k+1] - \mathbb{S}$ with the largest singleton $k+1$ and π' is the singletons of π_1 greater than $k+1$. Then the pair $(\pi, \mathbb{S})$ lies in $\mathbb{X}_{n,k}$.

Clearly we find a bijection between $\mathbb{X}_{n,k}$ and $\mathbb{Y}_{n,k}$, which proves (3.7). □

Setting $y=0$ and $y=1$ in (3.7), by (2.5) in the case $k=1$, we have

Corollary 3.7. *For any integers $n, k \geq 0$, there hold*

$$\begin{aligned} B_{n+k} &= \sum_{j=0}^{n} \binom{n}{j} A_{k+j,k}, \\ A_{n+k+1,n+1} &= \sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} 2^{n-j}. \end{aligned}$$

Corollary 3.8. *For any integers $n, k, m, i \geq 0$, there hold*

$$\sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} (n-j)! = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} D_{n-j}, \tag{3.10}$$

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} A_{k+j,k} A_{m+i+j,m} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} A_{n+m+i,n+m-j} B_{k+j}, \tag{3.11}$$

where D_n is the number of permutations of $[n]$ without fixed points.

Proof. The exponential generating function [19] for D_n is

$$\sum_{n\geq 0} D_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x},$$

from which, one can get

$$n! = \sum_{j=0}^{n} \binom{n}{j} D_{n-j}.$$

Let $\mathbf{D}$ be the umbra, given by $\mathbf{D}^n = D_n$, we have $n! = (\mathbf{D}+1)^n$. Then (3.10) can be obtained by setting $y = \mathbf{D}$ in (3.7).

Setting $y = \frac{y}{1-y}$ in (3.7) and multiplying $y^m (y-1)^{n+i}$ by the two sides, we have

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} A_{k+j,k} y^m (y-1)^{i+j} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{k+j} y^{n+m-j} (y-1)^{i+j},$$

which, when $y = \mathbf{B}$, yields (3.11). □

Gould and Quaintance [9] present the identity

$$\sum_{j=0}^{m} s(m,j)B_{k+j} \quad = \quad \sum_{i=0}^{k} \binom{k}{i} m^{k-i} B_i,$$

which is a special case ($n = 0$) of the following three identities.

Theorem 3.9. *For any integers $n, m, k \geq 0$, there hold*

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} \quad = \quad \sum_{i=0}^{k}\sum_{r=0}^{n} \binom{k}{i}\binom{n}{r} m^{n+k-i-r} A_{r+i,i}, \tag{3.12}$$

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} \quad = \quad \sum_{i=0}^{k}\sum_{r=0}^{n} \binom{k}{i}\binom{n}{r} m^{k-i}(m-1)^{n-r} B_{r+i}, \tag{3.13}$$

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j+1,n+1} \quad = \quad \sum_{i=0}^{k}\sum_{r=0}^{n} \binom{k}{i}\binom{n}{r} m^{k-i}(m+1)^{n-r} B_{r+i}, \tag{3.14}$$

where $s(k,j)$ are the first kind of Stirling numbers.

Proof. We know the Bell umbra $\mathbf{B}$ satisfies $\mathbf{B}^{m+1} = (\mathbf{B}+1)^m$. Then by linearity, for any polynomial $f(x)$ we have

$$\mathbf{B} f(\mathbf{B}) = f(\mathbf{B}+1),$$

which, by induction on integer $m \geq 0$, leads to

$$\mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-m+1)f(\mathbf{B}) = f(\mathbf{B}+m). \tag{3.15}$$

It is well known that for any indeterminant x,

$$x(x-1)\cdots(x-m+1) = \sum_{j=0}^{m} s(m,j)x^j,$$

Using the umbral representation for $A_{n,k}$, we have

$$\begin{aligned}
\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} \quad &= \quad \sum_{j=0}^{m} s(m,j)\mathbf{B}^{k+j}(\mathbf{B}-1)^n \\
&= \quad \mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-m+1)\mathbf{B}^k(\mathbf{B}-1)^n \\
&= \quad (\mathbf{B}+m)^k(\mathbf{B}-1+m)^n \\
&= \quad \sum_{i=0}^{k}\sum_{r=0}^{n} \binom{k}{i}\binom{n}{r} m^{n+k-i-r}\mathbf{B}^i(\mathbf{B}-1)^r \\
&= \quad \sum_{i=0}^{k}\sum_{r=0}^{n} \binom{k}{i}\binom{n}{r} m^{n+k-i-r} A_{r+i,i},
\end{aligned}$$

which proves (3.12). Similarly, one can deduce (3.13) and (3.14). □

Theorem 3.10. *For any integer $n \geq 0$, there hold*

$$\sum_{k=0}^{n}(k+1)A_{n,k} = (n+2)B_{n+1} - V_{n+3}, \tag{3.16}$$

$$\sum_{k=0}^{n}(n-k+1)A_{n,k} = V_{n+3} - (n+2)V_{n+1}. \tag{3.17}$$

Proof. Define

$$\alpha_n(x) = \sum_{k=0}^{n} A_{n,k}x^k.$$

By (2.3), we have

$$\begin{aligned}
\sum_{k=0}^{n} A_{n,k}x^k &= \frac{1}{e}\sum_{k=0}^{n} x^k \sum_{m=0}^{\infty} \frac{m^k(m-1)^{n-k}}{m!} \\
&= \frac{1}{e}\sum_{m=0}^{\infty} \frac{1}{m!}\sum_{k=0}^{n}(mx)^k(m-1)^{n-k} \\
&= \frac{1}{e}\sum_{m=0}^{\infty} \frac{1}{m!}\frac{(mx)^{n+1}-(m-1)^{n+1}}{mx-m+1}.
\end{aligned}$$

Differentiating $x\alpha_n(x)$ and then setting $x = 1$ gives

$$\begin{aligned}
\sum_{k=0}^{n}(k+1)A_{n,k} &= \frac{1}{e}\sum_{m=0}^{\infty} \frac{(n+1)m^{n+1} - m^{n+1}(m-1) + (m-1)^{n+2}}{m!} \\
&= (n+1)B_{n+1} - A_{n+2,n+1} + A_{n+2,0} \\
&= (n+1)B_{n+1} - (A_{n+2,n+2} - A_{n+1,n+1}) + (B_{n+2} - A_{n+3,0}) \\
&= (n+1)B_{n+1} - (B_{n+2} - B_{n+1}) + (B_{n+2} - V_{n+3}) \\
&= (n+2)B_{n+1} - V_{n+3},
\end{aligned}$$

which proves (3.16). Similarly, differentiating $x^{n+1}\alpha_n(x^{-1})$ and then setting $x = 1$ gives (3.17). □

Remark 3.11. *Canfield* [3] *has shown that the average number of singletons in a partition of $[n]$ is an increasing function of n. We guess that the average number of the largest or smallest singletons in a partition of $[n+1]$ is also an increasing function of n. That is to say, both*

$$\frac{(n+2)B_{n+1} - V_{n+3}}{B_{n+1}} \quad \text{and} \quad \frac{V_{n+3} - (n+2)V_{n+1}}{B_{n+1}}$$

are increasing functions of n. One can be asked for asymptotic formulas for the above two expressions.

4. Congruence properties of $A_{n,k}$ and Bell numbers B_n

In this section, based on umbral calculus, we study the congruence properties of $A_{n,k}$ and Bell numbers B_n. Throughout this section, p refers to a prime, and unless stated otherwise, all congruences are modulo p.

Theorem 4.1. *For any integers $n, m, k \geq 0$, there holds*

$$A_{n+pm+k,k} \equiv A_{n+m+k,m+k}.$$

Proof. Recall that the Lagrange congruence

$$x(x-1)\cdots(x-p+1) \equiv x^p - x,$$

and the binomial congruence

$$(x-1)^p \equiv x^p - 1.$$

Setting $x = \mathbf{B}$, by (3.15), for any polynomial $f(x)$, one gets

$$(\mathbf{B}^p - \mathbf{B})f(\mathbf{B}) \equiv \mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-p+1)f(\mathbf{B}) = f(\mathbf{B}+p) \equiv f(\mathbf{B}),$$

which, by induction on integer $j \geq 0$, leads to

$$(\mathbf{B}^p - \mathbf{B})^j f(\mathbf{B}) \equiv f(\mathbf{B}), \tag{4.1}$$

$$(\mathbf{B}-1)^{pj} f(\mathbf{B}) \equiv \mathbf{B}^j f(\mathbf{B}). \tag{4.2}$$

Using the umbral representation for $A_{n,k}$, we have

$$\begin{aligned} A_{n+pm+k,k} &= \mathbf{B}^k(\mathbf{B}-1)^{n+pm} \\ &= (\mathbf{B}-1)^{pm}\mathbf{B}^k(\mathbf{B}-1)^n \\ &\equiv \mathbf{B}^{m+k}(\mathbf{B}-1)^n \\ &= A_{n+m+k,m+k}, \end{aligned}$$

as claimed. □

Corollary 4.2. *For any integers $n, m \geq 0$, there hold*

$$B_{n+pm} \equiv A_{n+m+1,m+1}, \tag{4.3}$$

$$B_{n+p} \equiv B_n + B_{n+1}, \quad \textit{(Touchard's congruence [20, 21])}, \tag{4.4}$$

$$A_{(n+1)p,p} \equiv B_n + B_{n+1}, \tag{4.5}$$

$$B_{np} \equiv B_{n+1}, \quad \textit{(Comtet's congruence [4, 8])}. \tag{4.6}$$

Proof. The case $k = 1$ in Theorem 4.1 leads to (4.3), which in the case $m = 1$ yields (4.4). (4.5) follows by setting $n = 0, m = n, k = p$, and (4.6) follows by setting $n = 0, m = n, k = 1$ in Theorem 4.1. □

Theorem 4.3. *For any integers $n, m, k \geq 0$, there holds*

$$A_{n+p^m+k,k} \equiv mA_{n+k,k} + A_{n+k+1,k}.$$

Proof. By (4.1) and (4.2), when $f(x) = 1$, one has

$$\begin{aligned} \mathbf{B}^p &\equiv \mathbf{B} + 1, \\ (\mathbf{B}-1)^p &\equiv \mathbf{B}. \end{aligned}$$

Using the little Fermat's congruence $k^p \equiv k$, where k is an integer, by induction on integer $m \geq 0$, we have

$$(\mathbf{B}-1)^{p^m} \equiv \mathbf{B} + m - 1. \tag{4.7}$$

Then

$$\begin{aligned} A_{n+p^m+k,k} &= (\mathbf{B}-1)^{p^m}\mathbf{B}^k(\mathbf{B}-1)^n \\ &\equiv (\mathbf{B}+m-1)\mathbf{B}^k(\mathbf{B}-1)^n \\ &= m\mathbf{B}^k(\mathbf{B}-1)^n + \mathbf{B}^k(\mathbf{B}-1)^{n+1} \\ &= mA_{n+k,k} + A_{n+k+1,k}, \end{aligned}$$

as desired. □

Theorem 4.4. *Let $N_p = \frac{p^p-1}{p-1}$, for any integers $n, k \geq 0$, there hold*

$$\begin{aligned} A_{n+N_p+k,k} &\equiv A_{n+k,k}, \\ A_{n+N_p+k,N_p+k} &\equiv A_{n+k,k}, \end{aligned}$$

namely, the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ (mod p) both have the period N_p.

Proof. By (4.7) and the Lagrange cogruence, one has

$$(\mathbf{B}-1)^{N_p} = \prod_{j=1}^{p}(\mathbf{B}-1)^{p^{p-j}} \equiv \prod_{j=1}^{p}(\mathbf{B}-j-1) \equiv \prod_{j=0}^{p-1}(\mathbf{B}-j) \equiv 1.$$

Then

$$A_{n+N_p+k,k} = (\mathbf{B}-1)^{N_p}\mathbf{B}^k(\mathbf{B}-1)^n \equiv \mathbf{B}^k(\mathbf{B}-1)^n = A_{n+k,k}.$$

When $m = N_p$ in Theorem 4.1, one has

$$A_{n+N_p+k,N_p+k} \equiv A_{n+pN_p+k,k} \equiv A_{n+k,k},$$

where the last modular equation follows by the periodicity of $(A_{n+k,k})_{n\geq 0}$. □

Remark 4.5. *Hall showed that the Bell numbers (the case $k = 1$ for $(A_{n+k,k})_{n\geq 0}$ or the case $n = 0$ for $(A_{n+k,k})_{k\geq 0}$) have the period N_p, a result rediscovered by Williams* [22]. *Williams also showed that the minimum period is precisely N_p for $p = 2, 3$ and 5. Radoux* [13] *conjectured that N_p is the minimal period of the sequence B_n for any prime p. Levine and Dalton* [6] *showed that the minimum period is exactly N_p for $p = 7, 11, 13$ and 17. They also investigated the period for the other primes < 50. Recently, Montgomery, Nahm and Wagstaff* [12] *showed that the minimum period is exactly N_p for most primes p below 180.*

For the sequences$(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$, we also have the following conjecture.

Conjecture 4.6. *For any integer $k \geq 0$ and any prime p, the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ both have the minimum period N_p modulo p.*

Theorem 4.7. *Let $n, m, k \geq 0$ be integers and p be a prime. Then a necessary and sufficient condition that $A_{n+m+k,k} \equiv 0$ (mod p) for $m = 0, 1, \ldots, p-2$, is that $A_{n+m+k,k} \equiv A_{n+pm+k,k}$ (mod p) for $m = 1, 2, \ldots, p-1$.*

Proof. By Theorem 4.1 and (2.5), we have

$$A_{n+pm+k,k} \equiv A_{n+m+k,m+k} = \sum_{j=0}^{m}\binom{m}{j}A_{n+k+j,k}. \tag{4.8}$$

Therefore, if $A_{n+m+k,k} \equiv 0$ (mod p) for $m = 0, 1, \ldots, p-2$, we clearly have $A_{n+pm+k,k} \equiv 0$ and hence, trivially, $A_{n+pm+k,k} \equiv A_{n+m+k,k}(\equiv 0)$. When $m = p-1$ and $A_{n+j+k,k} \equiv 0$ for $j = 0, 1, \ldots, p-2$, (4.8) reduces to $A_{n+(p-1)+k,k} \equiv A_{n+p(p-1)+k,k}$.

Conversely, if $A_{n+m+k,k} \equiv A_{n+pm+k,k}$ (mod p) for $m = 1, 2, \ldots, p-1$, (4.8) is equivalent to

$$A_{n+m+k,k} \equiv A_{n+m+k,k} + \sum_{j=0}^{m-1}\binom{m}{j}A_{n+k+j,k}, \ (m = 1, 2, \ldots, p-1),$$

which reduces to

$$0 \equiv \sum_{j=0}^{m-1}\binom{m}{j}A_{n+k+j,k}, \ (m = 1, 2, \ldots, p-1). \tag{4.9}$$

The system (4.9) is triangular with diagonal coefficients $\binom{m}{m-1}$. The coefficient matrix is therefore nonsingular with determinant $(p-1)! \equiv -1$ by Wilson's theorem. Thus the only solution is given by $A_{n+m+k,k} \equiv 0$ (mod p) for $m = 0, 1, \ldots, p-2$. □

Theorem 4.8. *For any integer $k \geq 0$ and any prime p, there exists an integer $M_{p,k} \geq 0$ such that*

$$A_{M_{p,k}+m+k,k} \equiv 0, \ (0 \leq m \leq p-2),$$

where

$$M_{p,k} \equiv 1-(k-1)p-\frac{p^p-p}{(p-1)^2}, \qquad (mod\ N_p).$$

In other words, the sequence $(A_{n+k,k})_{n\geq 0}$ (mod p) contains a string of $p-1$ consecutive zeroes.

Proof. By Theorem 4.1 and (4.3), we have

$$A_{(n+(k-1)p-k)p+k,k} \equiv A_{n+k,k},$$

which, when $n = M_{p,k}+m$, where $M_{p,k}$ is an integer to be determined, produces

$$A_{(M_{p,k}+(k-1)p-k)p+pm+k,k} \equiv A_{M_{p,k}+m+k,k}.$$

By Theorem 4.4 and 4.7, it follows that $p-1$ consecutive zeros of $(A_{n+k,k})_{n\geq 0}$ (mod p) will occur, beginning with $A_{M_{p,k}+k,k}$, if there holds

$$A_{(M_{p,k}+(k-1)p-k)p+pm+k,k} \equiv A_{(M_{p,k}+(k-1)p-k)p+m+k,k}, \ (m=1,2,\ldots,p-1).$$

It is just required that the following condition holds

$$(M_{p,k}+(k-1)p-k)p+m \equiv M_{p,k}+m, \qquad (\text{mod } N_p),$$

or, equivalently, if there holds

$$(M_{p,k}+(k-1)p-k)p = M_{p,k}+rN_p,$$

for some integer r. Using $N_p = \frac{p^p-1}{p-1} = \frac{p^p-p}{p-1}+1$, routine calculation yields

$$M_{p,k} = 1-(k-1)p+\frac{r+1}{p-1}+r\frac{p^p-p}{(p-1)^2},$$

It is easy to verify by the binomial congruence that $\frac{p^p-p}{(p-1)^2}$ is always an integer. Since $M_{p,k}$ is also an integer, so we must have $r = -1+t(p-1)$ for some integer t, from which it follows that

$$M_{p,k} = 1-(k-1)p-\frac{p^p-p}{(p-1)^2}+tN_p.$$

Since the $p-1$ consecutive zeros start with $A_{M_{p,k}+k,k}$, the proof is complete. □

Using the same arguments, we have analogous results for the sequences $(A_{n+k,n})_{n\geq 0}$, their proofs are left to interested readers, the critical step for Theorem 4.10 is to show the congruence relation

$$A_{(n-1)p-k(p^{p-1}-1)+k,(n-1)p-k(p^{p-1}-1)} \equiv A_{n+k,n}.$$

Theorem 4.9. *Let $n, m, k \geq 0$ be integers and p be a prime. Then a necessary and sufficient condition that $A_{n+m+k,n+m} \equiv 0$ (mod p) for $m = 0,1,\ldots,p-2$, is that $A_{n+m+k,n+m} \equiv A_{n+pm+k,n+pm}$ (mod p) for $m = 1,2,\ldots,p-1$.*

Theorem 4.10. *For any integer $k \geq 0$ and any prime p, there exists an integer $U_{p,k} \geq 0$ such that*

$$A_{U_{p,k}+m+k,U_{p,k}+m} \equiv 0, \ (0 \leq m \leq p-2),$$

where

$$U_{p,k} \equiv 1+\frac{(p^{p-1}-1)k}{p-1}-\frac{p^p-p}{(p-1)^2}, \qquad (mod\ N_p).$$

In other words, the sequence $(A_{n+k,n})_{n\geq 0}$ *(mod p) contains a string of* $p-1$ *consecutive zeroes.*

Remark 4.11. *Radoux* [13] *shows that if the period of the residues of the Bell sequence* B_n *is equal to* N_p *for a given prime* p*, then there exists a number of* c*, depending on* p*, such that* $B_{c+m} \equiv 0$ *(mod p) for* $0 \leq m \leq p-2$*. He also obtains the location of such a string of consecutive zeros. Kahale* [10] *and Layman* [11] *show respectively by two entirely different methods that this result holds without the hypothesis that* N_p *is the minimal period. Their result is a special case of Theorem 4.8 for* $k=1$ *or of Theorem 4.10 for* $k=0$*. Our proof methods are similar to Layman's.*

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